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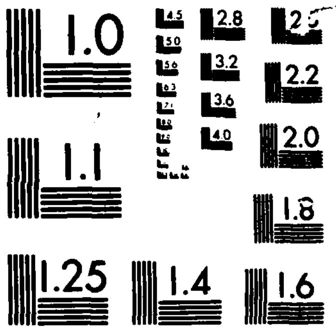

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Bounds for System Reliability Function

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Abstract

A coherent system with independent non-renewable components with arbitrary lifetime distributions is considered. A simple observation leads to a hierarchy of upper and lower bounds that converge to the exact system reliability. The simplest of these bounds is shown to be tighter than the bounds of Gertsbakh (1985).

1. Introduction

Consider a coherent system consisting of n independent non-renewable components $\{1, 2, .., n\}$, with lifetime distributions F_i, $i = 1, 2, .., n$. Let $X = \{\underline{X}(t), t \geq 0\}$ be the vector performance process of these components such that $\underline{X}(t) = (X_1(t), X_2(t), .., X_n(t))$ and $X_i(t)$ takes the value 0 if component i is up at time t and 1 otherwise. The state space S of X is then $\{0, 1\}^n$. Let ϕ be the coherent structure function of the system. That is ϕ:S $\rightarrow \{0, 1\}$, is non-decreasing on S and when $\phi(\underline{X}(t))$ is 0 the system functions at time t and is 1 otherwise. The reliability R(t) of the system at time t is then given by $1 - E\{\phi(\underline{X}(t))\}$. It is of interest to obtain R. In general it is not possible to obtain explicit formula for R and one resort to algorithmic methods to compute R. Except in some special cases [e.g. Agrawal and Satyanarayana (1984), Provan and Ball (1984), and Shanthikumar (1982, 1984)], the problem is in general NP-hard [e.g. Buzacott (1980), and Ball (1984)]. Consequently approximation and bounds have been developed [e.g. Ball and Provan (1983), Gertsbakh (1985), Shanthikumar (1984), and Shogan (1976)]. In this paper we derive alternative bounds for R and compare them to that of Gertsbakh (1985).

Let $h(\underline{p}) \equiv E\{\phi(\underline{X}(t))\}$, where $p_i \equiv E\{X_i(t)\} = F_i(t)$, and let (G, B) be a partition of S such that $G = \{\underline{x}:\phi(\underline{x}) = 0, \underline{x} \in S\}$ and define $U_\ell = \{\underline{x}: ||\underline{x}|| = \ell, \underline{x} \in S\}, \ell = 1, 2, .., n,$ where $||\underline{x}|| \equiv \sum x_i$. Suppose $\min\{||\underline{x}||: \underline{x} \in B\} = r$. Then $B = \bigcup_{j=r} V_j$, where $V_j = B \cap U_j^{i-1}, j = r, r+1, .., n$. With these observations one has,

$$h(\underline{p}) = \sum_{\underline{x} \in B} \left\{ \prod_{i \in \overline{W}(\underline{x})} p_i \right\} \left\{ \prod_{j \in W(\underline{x})} (1 - p_j) \right\} , \tag{1}$$

$$= \sum_{\ell = r}^{n} A_\ell , \tag{2}$$

where

$$A_\ell = \sum_{\underline{x} \in V_\ell} \left\{ \prod_{i \in \overline{W}(\underline{x})} p_i \right\} \left\{ \prod_{j \in W(\underline{x})} (1 - p_j) \right\} , \tag{3}$$

$W(\underline{x}) = \{k: x_k = 0, k = 1, 2, .., n\}$, is the set of working components in state $\underline{x}$, and $\overline{W}(\underline{x}) = \{1, 2, .., n\} - W(\underline{x})$.

Suppose we have upper and lower bounds $B_\ell \geq A_\ell \geq C_\ell$, $\ell = r, r+1, .., n$. Then we will have a sequence of upper and lower bounds for $h(\underline{p})$ and consequently for R. Specifically

$$\sum_{\ell=r}^{m-1} A_\ell + \sum_{\ell=m}^{n} C_\ell \le h(\underline{p}) \le \sum_{\ell=r}^{m-1} A_\ell + \sum_{\ell=m}^{n} B_\ell \ , \tag{4}$$

for m=r,r+1,..n+1, [we use $\sum_a^b \gamma_i = 0, b<a$ for all γ_i]. Note that the above bounds get tighter as m increases and give the exact unrealibility when m = n + 1. This allows one to progressively compute $A_r, A_{r+1}, \ldots$, until a desired accuracy is found.

In Section 2 we develop some upper and lower bounds for A_ℓ and compare it to that of Gertsbakh (1985). A simple numerical example is given in Section 3.

2. The Bounds

Without loss of generality assume that the components are numbered so that $p_1 \ge p_2 \ge \ldots \ge p_n$. Since $p_i = F_i(t)$ depend on t, this numbering may have to be altered for different values of t, unless $F_1(t) \ge F_2(t) \ge \ldots \ge F_n(t)$, for all $t \ge 0$.

Proposition 2.1: Let $||V_\ell||$ be the number of elements in the set V_ℓ. Then

$$||V_\ell|| \left\{ \prod_{i=1}^{\ell} p_{n+1-i} \right\} \left\{ \prod_{j=\ell+1}^{n} (1-p_{n+1-j}) \right\} \equiv C_\ell^{(1)}$$

$$\le A_\ell \le ||V_\ell|| \left\{ \prod_{i=1}^{\ell} p_i \right\} \left\{ \prod_{j=\ell+1}^{n} (1-p_j) \right\} \equiv B_\ell^{(1)}. \tag{5}$$

Proof: It is easily seen that for any $\underline{x} \in V_\ell$,

$$\left\{ \prod_{i=1}^{\ell} p_{n+1-i} \right\} \left\{ \prod_{j=\ell+1}^{n} (1-p_{n+1-j}) \right\} \le \left\{ \prod_{i \in \bar{W}(\underline{x})} p_i \right\} \left\{ \prod_{j \in W(\underline{x})} (1-p_j) \right\}$$

$$\le \left\{ \prod_{i=1}^{\ell} p_i \right\} \left\{ \prod_{j=\ell+1}^{n} (1-p_j) \right\}.$$

The result now directly follows from (3).

$\square$

To use the bounds $C_\ell^{(1)}$ and $B_\ell^{(1)}$ one needs to know $||V_\ell||$. Once $||V_\ell||$, $\ell = r, r+1, \ldots$, are known one may use the recursion $B_{\ell+1}^{(1)} = \{||V_{\ell+1}|| / ||V_\ell||\} \{p_{\ell+1}/(1-p_{\ell+1})\}$ to compute $B_\ell^{(1)}, \ell = r, r+1, \ldots$ [A similar recursion may be used for $C_\ell^{(1)}, \ell = r, r+1, \ldots$] In some cases obtaining $||V_\ell||$ itself may turn out to be computationally difficult. In such a case the observation that $||V_\ell|| \le \binom{n}{\ell}$ leads to a modification of $B_\ell^{(1)}$ as follows:

Proposition 2.2:

$$A_\ell \le \binom{n}{\ell} \left\{ \prod_{i=1}^{\ell} p_i \right\} \left\{ \prod_{j=\ell+1}^{n} (1-p_j) \right\} \equiv B_\ell^{(2)}. \tag{6}$$

We will next obtain a modification of $B_\ell^{(2)}$. Define $P_i^+ = \sum_{j=1}^{n+1-i} p_j/(n+1-i)$ and $P_i^- = \sum_{j=i} p_j/(n+1-i), i = 1, 2, \ldots, n$. For any $\underline{y} \in S$ let

$$Q_i(\underline{y}) = \begin{cases} P_i^+ & \text{if } y_i = 1 \\ 1 - P_i^- & \text{if } y_i = 0 \end{cases}$$

Then

Proposition 2.3: For any $\underline{y} \in U_\ell$,

$$A_\ell \le \binom{n}{\ell} \prod_{i=1}^{n} Q_i(\underline{y}). \tag{7}$$

Proof: Let P be the collection of all permutations of $\{1, 2, \ldots, n\}$. Then

$$\sum_{\underline{x} \in U_\ell} \left\{ \prod_{i \in \overline{W}(\underline{x})} p_i \right\} \left\{ \prod_{j \in W(\underline{x})} (1-p_j) \right\}$$

$$= \frac{1}{\ell!(n-\ell)!} \left\{ \sum_{\pi \in P} \prod_{i=1}^{n} q_i(\pi, \underline{y}) \right\} \tag{8}$$

where

$$q_i(\pi, \underline{y}) = \begin{cases} p_{\pi(i)} & \text{if} \quad y_i = 1 \\ 1 - p_{\pi(i)} & \text{if} \quad y_i = 0 \ . \end{cases}$$

Now consider a subset P^{n-1} of P, such that the first (n-1) elements in each of the permutations in P^{n-1} are identical. Then for any $\pi' \in P^{n-1}$

$$\sum_{\pi \in P^{n-1}} \prod_{i=1}^{n} q_i(\pi, \underline{y}) = \prod_{i=1}^{n-1} q_i(\pi', \underline{y}) \sum_{\pi \in P^{n-1}} q_n(\pi, \underline{y})$$

$$\leq \prod_{i=1}^{n-1} q_i(\pi', \underline{y}) \sum_{\pi \in P^{n-1}} Q_n(\underline{y})$$

$$= \sum_{\pi \in P^{n-1}} \prod_{i=1}^{n-1} q_i(\pi, \underline{y}) Q_n(\underline{y}),$$

since by definition $Q_n(\underline{y}) \geq q_n(\pi, \underline{y})$ for all $\pi \in P^{n-1}$. Note that $||P^{n-1}|| = 1$. Therefore

$$\sum_{\pi \in P} \prod_{i=1}^{n} q_i(\pi, \underline{y}) \leq \sum_{\pi \in P} \prod_{i=1}^{n-1} q_i(\pi, \underline{y}) Q_n(\underline{y}).$$

Now defining P^{n-2} to be a subset of P such that the first n-2 elements of every permutation in there is identical one sees that

$$\sum_{\pi \in P^{n-2}} \prod_{i=1}^{n-2} q_i(\pi, \underline{y}) \leq \sum_{\pi \in P^{n-2}} \prod_{i=1}^{n-2} q_i(\pi, \underline{y}) Q_{n-1}(\underline{y}),$$

since by definition $2.Q_{n-1}(\underline{y}) \geq \sum_{\pi \in P^{n-2}} q_{n-1}(\pi, \underline{y})$. Note that $||P^{n-2}|| = 2$. Then

$$\sum_{\pi \in P} \prod_{i=1}^{n} q_i(\pi, \underline{y}) \leq \sum_{\pi \in P} \prod_{i=1}^{n-2} q_i(\pi, \underline{y}) \prod_{j=n-1}^{n} Q_j(\underline{y}).$$

Continuing this way and observing that $||P|| = n!$, Equation (8) leads to the required result.

□

Since (7) may give different upperbounds for different $\underline{y}$'s, it is appropriate that we choose the minimum of such bounds. In this regard define π^* to be a permutation of $\{1,2,..,n\}$ such that

$$\frac{P^+_{\pi^*(1)}}{1-P^-_{\pi^*(1)}} \le \frac{P^+_{\pi^*(2)}}{1-P^-_{\pi^*(2)}} \le .... \le \frac{P^+_{\pi^*(n)}}{1-P^-_{\pi^*(n)}}. \tag{9}$$

Then using a straightforward pairwise interchange it can be shown that

Proposition 2.4

$$\min_{\underline{y}\in U_\ell}\left\{\prod_{i=1}^{n} Q_i(\underline{y})\right\}$$

$$=\prod_{i=1}^{\ell} P^+_{\pi^*(i)} \prod_{j=\ell+1}^{n}(1-P^-_{\pi^*(j)}).$$

Equivalently

$$A_\ell \le \binom{n}{\ell}\prod_{i=1}^{\ell} P^+_{\pi^*(i)} \prod_{j=\ell+1}^{n}(1-P^-_{\pi^*(j)}) \equiv B^{(3)}_\ell \tag{10}$$

We will next provide an alternative upper bound for $\sum_{\ell=k}^{n} A_\ell$ which is tighter than that given by Gertsbakh (1985).

Proposition 2.5:

$$\sum_{\ell=k}^{n} A_\ell \le \binom{n}{k}\prod_{i=1}^{k} P^+_i = \sum_{\ell=k}^{n} B^{(4)}_\ell \tag{11}$$

Proof: Observe that

$$\sum_{\ell=k}^{n} A_\ell \le \mathrm{P}\{\text{at least k components are down at time t}\}. \tag{12}$$

Since for any $\pi \in P$,

$$P\{\text{components } \pi(1), \pi(2), \ldots, \pi(k) \text{ are down}\} = \prod_{i=1}^{k} p_{\pi(i)},$$

one has from (12)

$$\sum_{l=k}^{n} A_l \leq \frac{1}{k!(n-k)!} \sum_{\pi \in P} \prod_{i=1}^{k} p_{\pi(i)}. \tag{13}$$

Using an analysis similar to that used for the proof of Proposition 2.3 one gets

$$\sum_{\pi \in P} \prod_{i=1}^{k} p_{\pi(i)} \leq n! \prod_{i=1}^{k} P_i^+. \tag{14}$$

Combining (13) and (14) one obtains the desired result.

□

When the lifetime of component i is exponentially distributed with rate λ_i, one has $p_i = 1 - \exp(-\lambda_i t)$, $i = 1,2,\ldots,n$. With $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$,

$$P_i^+ = \frac{1}{n+1-i} \sum_{j=1}^{n+1-i} (1 - \exp(-\lambda_j t))$$

$$\leq \frac{1}{n+1-i} \Lambda_{i-1} t.$$

where $\Lambda_i \equiv \sum_{j=1}^{n-i} \lambda_j$. Therefore from (11), one gets

$$\sum_{l=r+1}^{n} A_l \leq \binom{n}{r+1} \prod_{i=1}^{r+1} P_i^+$$

$$\leq \binom{n}{r+1} \prod_{i=0}^{r} \left\{ \frac{\Lambda_i t}{n-i} \right\}$$

$$= \left(\prod_{i=0}^{r} \Lambda_i \right) \frac{t^{r+1}}{(r+1)!} \tag{15}$$

The right hand side of (15) is the bound obtained by Gertsbakh (1985). Clearly (11) is a tighter bound and is applicable to any component lifetime distributions.

In summary, one can use (4) along with (5), (6), (10), and (11) to obtain alternative bounds for the system reliability function R. Since the computation of these bounds are relatively simple, all these bounds may be computed and the best value can be used.

3. A Numerical Example

Consider the network shown below in Figure 1. Suppose we are interested in the reliability that node s is connected to node t by at least one path of working edges.

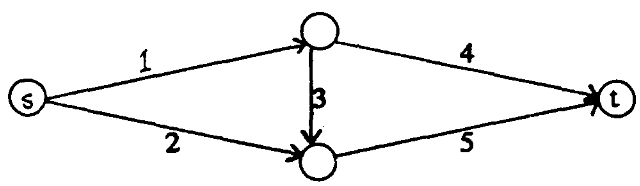

Figure 1: Directed Reliability Network

i	1	2	3	4	5
p_i	.1	.1	.08	.08	.08

Table 1: Edge unreliabilities (p_i).

The edge failure probabilities $p_i = 1, 2, 3, 4, 5$ are given in Table 1. Then the minimum number of failed edges needed for a cut-set is r=2, and

$$V_2 = \{(0,0,1,1,1), (1,1,1,0,0), (0,1,1,1,0)\}$$

$$V_3 = \{(0,0,0,1,1), (0,0,1,0,1), (0,0,1,1,0),$$

$$(0,1,1,0,0),(1,0,1,0,0),(1,1,0,0,0),$$

$$(0,1,0,1,0),(1,0,0,0,1,1)\}$$

$$V_4 = \{(0,0,0,0,1),(0,0,0,1,0),(0,0,1,0,0),(0,1,0,0,0),(1,0,0,0,0)\}$$

$$V_5 = \{(0,0,0,0,0)\}.$$

with $||V_2|| = 3$, $||V_3|| = 8$, $||V_4|| = 5$ and $||V_5|| = 1$. Exact values, upper and lower bounds of $A_\ell, \ell = 2,3,4,5$ are calculated using Equations (3), (5), (6), (10), and (11) and tabulated in Table 2. In this example P_i^+ is 0.088, 0.09, 0.09333, 0.1, 0.1 and P_i^- is 0.088, 0.085, 0.08, 0.08, 0.08 for i = 1, 2, 3, 4 and 5 respectively. Hence $\pi^* = <1,2,3,4,5>$, thus resulting in the same values for the bounds $B_\ell^{(2)}$ and $B_\ell^{(3)}$ [Equations (6) and (10)]. Substituting the upper and lower bounds for A_ℓ in Equation (4) one obtains the bounds for the unreliability $h(\underline{p})$ [shown in Table 3].

m	2	3	4	5
$Eq(3)\ \sum_{\ell=m}^{n} A_\ell$	0.02348992	0.00483968	0.00027392	0.00000512
$Eq(5)\ \sum_{\ell=m}^{n} C_\ell^{(1)}$	0.01786112	0.00355328	0.00023552	0.00000512
$Eq(5)\ \sum_{\ell=m}^{n} B_\ell^{(1)}$	0.02907712	0.00571648	0.00029952	0.00000512
$Eq(6)\ \sum_{\ell=m}^{n} B_\ell^{(2)}$	0.08493952	0.00707072	0.00029952	0.00000512
$Eq(10)\ \sum_{\ell=m}^{n} B_\ell^{(3)}$	0.08493952	0.00707072	0.00029952	0.00000512
$Eq(11)\ \sum_{\ell=m}^{n} B_\ell^{(4)}$	0.0792	0.007392	0.0003696	0.000007392

Table 2: Exact Values, Upper and Lower Bounds for $\sum_{\ell=m}^{n} A_\ell$.

m	Lower bound using Eq(5)	Upper bounds Using			
		Eq(5)	Eq(6)	Eq(10)	Eq(11)
2	0.01786112	0.02907712	0.08493952	0.08493952	0.0792
3	0.02220352	0.02436672	0.02572096	0.02572096	0.02604224
4	0.02345152	0.02351552	0.02351552	0.02351552	0.0235856
5	0.02348992*	0.02348992*	0.02348992*	0.02348992*	0.0234901472

* Exact unreliability.

Table 3: Upper and Lower Bounds for $h(\underline{p})$.

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